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A Basic Algebra Is an MV-Algebra If and Only If It Is a BCC-Algebra

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Abstract The aim of the paper is to show that every lattice with section antitone involutions, i.e. a lattice having antitone involutions on its principal filters, is an MV-algebra if and only if it is a BCC-algebra.

Keywords Lattice · Section antitone involution · MV-algebra · BCC-algebra · BCK-algebra

By an **MV-algebra** is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
(MV2) $x \oplus y = y \oplus x$
(MV3) $x \oplus 0 = x$
(MV4) $\neg\neg x = x$
(MV5) $x \oplus \neg 0 = \neg 0$
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

This concept was introduced by C.C. Chang [2] as an algebraic counterpart of many-valued Łukasiewicz logic, and the definition in present form is taken from [3].

As known, to each MV-algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ can be assigned the bounded distributive lattice $\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)$ where

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$$x \vee y = \neg(\neg x \oplus y) \oplus y,$$

$$x \wedge y = \neg(\neg x \vee \neg y),$$

$$1 = \neg 0$$

and whose induced order is defined by

(P) $\quad x \leq y \quad$ if and only if $\quad \neg x \oplus y = 1.$

We can generalize this concept as follows:

By a **basic algebra** we mean an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ fulfilling the identities:

(B1) $x \oplus 0 = x$
(B2) $\neg\neg x = x$
(B3) $x \oplus \neg 0 = \neg 0 = \neg 0 \oplus x$
(B4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(B5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$
(B6) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y.$

In the sequel, $\neg 0$ will be denoted by 1. Clearly, $\neg 1 = 0$.

Lemma 1 *Every basic algebra satisfies the identities*

$$0 \oplus x = x \quad \textit{and} \quad \neg x \oplus x = x \oplus \neg x = 1.$$

Proof When putting $y = 0$ in (B4) we get by (B2), (B1) and (B3)

$$x = \neg\neg x = \neg(\neg x \oplus 0) \oplus 0 = \neg(\neg 0 \oplus x) \oplus x = \neg(1 \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x.$$

Further, $\neg x \oplus x = \neg(0 \oplus x) \oplus x = \neg(\neg 1 \oplus x) \oplus x = \neg(\neg x \oplus 1) \oplus 1 = 1$.
Finally, (B2) yields $1 = \neg\neg x \oplus \neg x = x \oplus \neg x$. □

Theorem 1 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Define a binary relation $\leq$ on A by* (P). *Then $(A; \leq)$ is an ordered set such that* 0 *resp.* 1 *is its least resp. greatest element.*

Proof Obviously, $\leq$ is reflexive by Lemma 1. Assume $x \leq y$ and $y \leq x$. Then $\neg x \oplus y = 1$ and $\neg y \oplus x = 1$ thus

$$x = 0 \oplus x = \neg 1 \oplus x = \neg(\neg y \oplus x) \oplus x = \neg(\neg x \oplus y) \oplus y = \neg 1 \oplus y = 0 \oplus y = y$$

proving that $\leq$ is antisymmetric.

Suppose further that $x \leq y$ and $y \leq z$. Then $\neg x \oplus y = 1$, $\neg y \oplus z = 1$ and, applying (B5) we derive

$$\neg x \oplus z = \neg 1 \oplus (\neg x \oplus z) = \neg(\neg y \oplus z) \oplus (\neg x \oplus z) = \neg(\neg(\neg 1 \oplus y) \oplus z) \oplus (\neg x \oplus z)$$
$$= \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) = 1,$$

verifying $x \leq z$. Hence, $\leq$ is an order on A. Since $\neg 0 \oplus x = 1$ and $\neg x \oplus 1 = 1$ by (B3), we have $0 \leq x \leq 1$ for each $x \in A$. □

Lemma 2 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $\leq$ its induced order. Then*

(a) $x \leq y$ *if and only if* $\neg y \leq \neg x$
(b) $x \leq y$ *implies* $x \oplus z \leq y \oplus z$
(c) $y \leq x \oplus y$.

Proof Suppose $x \leq y$. Then $\neg x \oplus y = 1$ and, by (B5),

$$\neg(\neg y \oplus z) \oplus (\neg x \oplus z) = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) = 1$$

thus $\neg y \oplus z \leq \neg x \oplus z$. This yields immediately $\neg y \leq \neg x$ and, due to (B2), also the converse holds. Hence, $x \leq y$ implies also $x \oplus z \leq y \oplus z$. For the last assertion, we have $y = 0 \oplus y \leq x \oplus y$. □

Theorem 2 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $\leq$ its induced order. Then $(A; \leq)$ is a bounded lattice where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$.*

Proof Due to Lemma 2(c), $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ is a common upper bound of x, y with respect to $\leq$. Assume $x, y \leq z$. By Lemma 2(a) and (b) we have

$$\neg x \oplus y \geq \neg z \oplus y,$$

thus

$$\neg(\neg x \oplus y) \oplus y \leq \neg(\neg z \oplus y) \oplus y = \neg(\neg y \oplus z) \oplus z = 0 \oplus z = z$$

showing that $\neg(\neg x \oplus y) \oplus y$ is the least upper bound of x, y, i.e. $x \vee y = \neg(\neg x \oplus y) \oplus y$.
As an immediate consequence of Lemma 2(a) we obtain $x \wedge y = \neg(\neg x \vee \neg y)$. □

By a **lattice with section antitone involutions** is meant a system $\mathcal{L} = (L; \vee, \wedge\, (^a)_{a \in L}, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice such that for each $a \in L$ it possesses an **antitone involution** $x \mapsto x^a$ on the **section** $[a, 1]$, i.e. for $x, y \in [a, 1]$ it holds $x^{aa} = x$ and $x \leq y \Rightarrow y^a \leq x^a$.

Theorem 3 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $\vee, \wedge$ defined as in Theorem 2. For each $a \in A$ and $x \in [a, 1]$ define $x^a = \neg x \oplus a$. Then $\mathcal{L}(A) = (A; \vee, \wedge\, (^a)_{a \in L}, 0, 1)$ is a lattice with section antitone involutions.*

Proof Due to Theorem 2 we need only to show that for each $a \in A$ the mapping $x \mapsto x^a = \neg x \oplus a$ is an antitone involution on the interval $([a, 1]; \leq)$. Since $x \in [a, 1]$, we have $a \leq x$ and hence $x^{aa} = \neg(\neg x \oplus a) \oplus a = x \vee a = x$. If $x, y \in [a, 1]$ and $x \leq y$ then, by Lemma 2(a) and (b) we conclude $y^a = \neg y \oplus a \leq \neg x \oplus a = x^a$. □

Moreover, we can prove also the converse:

Theorem 4 *Let $\mathcal{L} = (L; \vee, \wedge\, (^a)_{a \in L}, 0, 1)$ be a lattice with section antitone involutions. Define*

$$x \oplus y = (x^0 \vee y)^y \quad \text{and} \quad \neg x = x^0.$$

Then $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ is a basic algebra. Moreover, $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$ and for any basic algebra $\mathcal{A}$ also $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$.

Proof Let $\mathcal{L} = (L; \vee, \wedge (^a)_{a\in L}, 0, 1)$ be a lattice with section antitone involutions. Then $\oplus$ is correctly defined since $x^0 \vee y \geq y$ thus really $x^0 \vee y \in [y, 1]$. We check the axioms (B1)–(B6):

(B1) $x \oplus 0 = (x^0 \vee 0)^0 = x^{00} = x$;
(B2) $\neg\neg x = x^{00} = x$;
(B3) $x \oplus 1 = (x^0 \vee 1)^1 = 1^1 = 1$ and $1 \oplus x = (1^0 \vee x)^x = (0 \vee x)^x = x^x = 1$;
(B4) Since $\neg x \oplus y = (x^{00} \vee y)^y = (x \vee y)^y \geq y$, we have
$\neg(\neg x \oplus y) \oplus y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$ and
$\neg(\neg y \oplus x) \oplus x = y \vee x = x \vee y$;
(B5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg(\neg(\neg x \vee y) \oplus z) \oplus (x \oplus z) = ((x^0 \vee y \vee z)^z \vee (x^0 \vee z)^z)^{(x^0 \vee z)^z} = ((x^0 \vee z)^z)^{(x^0 \vee z)^z} = 1$;
(B6) $\neg(\neg(x \oplus y) \oplus y) \oplus y = \neg(\neg x \vee y) \oplus y = (x^0 \vee y \vee y)^y = (x^0 \vee y)^y = x \oplus y$.

Hence $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ is a basic algebra. Let us denote by $\sqcup$ and $\sqcap$ the operation join and meet in the lattice $\mathcal{L}(\mathcal{A}(L))$ derived by Theorem 2 and by f_a (for $a \in L$) the involution in $[a, 1]$. Then we have for $x \in [a, 1]$

$$f_a(x) = \neg x \oplus a = (x^{00} \vee a)^a = (x \vee a)^a = x^a.$$

Further, $x \sqcup y = \neg(\neg x \oplus y) \oplus y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$ and $x \sqcap y = \neg(\neg x \sqcup \neg y) = (x^0 \vee y^0)^0 = x \wedge y$. Thus $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$.

Now let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Denote by $+$ and $\sim$ the additive operation and the negation in $\mathcal{A}(\mathcal{L}(A))$. Then $\sim x = x^0 = \neg x$ by (B1) and, applying (B6)

$$x + y = (x^0 \vee y)^y = \neg(\neg x \vee y) \oplus y = \neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y.$$

Thus also $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$. □

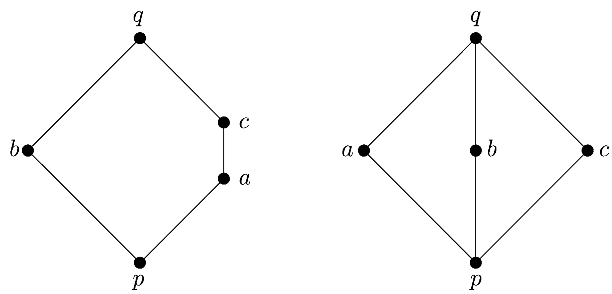
We say that a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is **commutative** if it satisfies the identity

$$x \oplus y = y \oplus x.$$

Theorem 5 *If the basic algebra* $\mathcal{A} = (A; \oplus, \neg, 0)$ *is commutative then the induced lattice* $\mathcal{L}(A)$ *is distributive.*

Proof Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra. We need to show that the lattice $\mathcal{L}(A)$ does not contain a sublattice isomorphic to N_5 or M_3 as visualized in Fig. 1.

Fig. 1



Since the mapping $x \mapsto x^p = \neg x \oplus p$ is an antitone involution on the interval $[p, 1]$, also $\{q^p, a^p, b^p, c^p, 1 = p^p\}$ forms a sublattice isomorphic to N_5 or M_3. Finally, the mapping $x \mapsto \neg x$ is an antitone involution on A thus $\{0 = \neg 1, \neg a^p, \neg b^p, \neg c^p, \neg q^p\}$ is again a sublattice isomorphic to N_5 or M_3.

In either case we have

$$a^p = 1^{a^p} = (b^p \vee a^p)^{a^p} = \neg b^p \oplus a^p = a^p \oplus \neg b^p = (\neg a^p \vee \neg b^p)^{\neg b^p} = (\neg q^p)^{\neg b^p}$$

and, when interchanging a and c, we get analogously $c^p = (\neg q^p)^{\neg b^p}$.
Hence $a^p = c^p$, i.e. $a = a^{pp} = c^{pp} = c$, a contradiction. □

The concept of BCC-algebra was introduced by Y. Komori [6] in connection with the problem whether the class of BCK-algebras (see [4, 5]) forms a variety. The problem was solved in negative. Recall that a **BCC-algebra** is an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2, 0)$ satisfying the axioms

(BCC1) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$

(BCC2) $x \to x = 1$

(BCC3) $x \to 1 = 1$

(BCC4) $1 \to x = x$

(BCC5) $(x \to y = 1$ and $y \to x = 1)$ implies $x = y$.

It is well-known that the binary relation $\leq$ defined by

(Q) $\quad x \leq y \quad$ if and only if $\quad x \to y = 1$

is an order on A with a greatest element 1. Hence, the identity (BCC1) can be rewritten as

(A) $\quad x \to y \leq (z \to x) \to (z \to y).$

The concept of **BCK-algebra** was introduced by Y. Imai and K. Iséki [4] as an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2, 0)$ satisfying the identities

(BCK1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1;$
(BCK2) $x \to ((x \to y) \to y) = 1$

and (BCC2)–(BCC5).
A BCK-algebra $\mathcal{A}$ is called
bounded if it has a least element with respect to the induced ordering;
commutative if it satisfies the identity $(x \to y) \to y = (y \to x) \to x$.

The following assertion is well known:

Proposition 1 *Every BCK-algebra is a BCC-algebra. A BCC-algebra is a BCK-algebra if and only if it satisfies the so-called Exchange Identity*

(EI) $\quad x \to (y \to z) = y \to (x \to z).$

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be an MV-algebra. Define $x \to y = \neg x \oplus y$. It was shown by D. Mundici [7] that for each MV-algebra $\mathcal{A}$, the algebra $\mathcal{B}(A) = (A; \to, \neg 0)$ is a commutative BCK-algebra and bounded commutative BCK-algebras are categorically equivalent to MV-algebras.

The following result was published in [1]:

Proposition 2 *Let $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ be a lattice with sectional antitone involutions. The following are equivalent*:

(1) *$\mathcal{A}(L)$ is an MV-algebra*
(2) *$\mathcal{L}$ is distributive and for $x \to y = (x \vee y)^y$ the Exchange Identity holds in $(L; \to, 1)$.*

Our aim is to show that for basic algebras the strong identity (EI) can be replaced by an essentially weaker identity (BCC1).

Given a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, similarly as for MV-algebras define $x \to y = \neg x \oplus y$. It is an easy exercise to check the axioms (BCC2)–(BCC5). Moreover, the order defined by (Q) coincides with the induced order of $\mathcal{A}$ (defined by (P)). Hence, a basic algebra $\mathcal{A}$ becomes a BCC-algebra if and only if it satisfies the identity (BCC1) which is equivalent to (A). Moreover, basic algebras satisfy the double negation law (B2) thus (A) is equivalent to

$$(A^*) \qquad x \oplus y \leq \neg(z \oplus \neg x) \oplus (z \oplus y).$$

Theorem 6 *A basic algebra is an MV-algebra if and only if it satisfies the identity (A^*), i.e. if it is a BCC-algebra.*

Proof If $\mathcal{A}$ is an MV-algebra then it is known to be a basic algebra. Since $\mathcal{A}$ is commutative, the lattice $\mathcal{L}(A)$ is distributive by Theorem 5 and, due to Proposition 2, it satisfies the Exchange Identity. Hence, by Proposition 1, it is a BCC-algebra.

Conversely, let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra satisfying the identity (A) (or equivalently (A^*) or (BCC1)) for $x \to y = \neg x \oplus y$.

Put $y = 0$ in (A^*) and replace x by $\neg x$. We get

$$\neg x = \neg x \oplus 0 \leq \neg(z \oplus x) \oplus z.$$

By Lemma 2(a) and (b) we obtain

$$x \oplus z \geq \neg(\neg(z \oplus x) \oplus z) \oplus z = (z \oplus x) \vee z \geq z \oplus x.$$

Interchanging x and z one can show

$$z \oplus x \geq x \oplus z,$$

thus $\mathcal{A}$ satisfies the identity $x \oplus z = z \oplus x$, i.e. $\mathcal{A}$ is commutative and, by Theorem 5, $\mathcal{L}(A)$ is distributive.

Since $\oplus$ is commutative, (A^*) is equivalent to

$$(A^{**}) \qquad \neg y \oplus x \leq \neg(\neg x \oplus z) \oplus (\neg y \oplus z).$$

Replacing x by $\neg x \oplus z$ in (A^{**}), we get

$$\neg y \oplus (\neg x \oplus z) \leq \neg(\neg(\neg x \oplus z) \oplus z) \oplus (\neg y \oplus z) = \neg(x \vee z) \oplus (\neg y \oplus z).$$

Hence, for $x, y \in [z, 1]$ we have

$$\neg y \oplus (\neg x \oplus z) \leq \neg x \oplus (\neg y \oplus z).$$

Analogously interchanging x and y we obtain

$$\neg x \oplus (\neg y \oplus z) \leq \neg y \oplus (\neg x \oplus z),$$

thus $\mathcal{A}$ satisfies for $x, y \in [z, 1]$ the equality

$$x \to (y \to z) = \neg x \oplus (\neg y \oplus z) = \neg y \oplus (\neg x \oplus z) = y \to (x \to z).$$

Since $a \to b = (a \vee b) \to b$, the Exchange Identity (EI) holds in $\mathcal{A}$ if and only if $\mathcal{A}$ satisfies

$$x \to (y \to z) = y \to (x \to z)$$

for $x, y \in [z, 1]$ (see (B6)). Hence, we have shown that $\mathcal{A}$ satisfies also (EI) and, due to Proposition 2 and Theorem 4, $\mathcal{A}$ is an MV-algebra. □

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